

THE TATE-HOCHSCHILD COHOMOLOGY RING OF A GROUP ALGEBRA

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ABSTRACT. We show that the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^*(RG, RG)$ of a finite group algebra RG is isomorphic to a direct sum of the Tate cohomology rings of the centralizers of conjugacy class representatives of G . Moreover, our main result provides an explicit formula for the cup product in $\widehat{\mathrm{HH}}^*(RG, RG)$ with respect to this decomposition. As an example, this formula helps us to compute the Tate-Hochschild cohomology ring of the symmetric group S_3 with coefficients in a field of characteristic 3.

1. INTRODUCTION

The theory of group cohomology is a well-studied yet ongoing research area. It has many applications to other areas such as representation theory, algebraic geometry, and commutative algebra. It is well-known that the Hochschild cohomology of a group algebra agrees with its usual cohomology, in the following sense. For an arbitrary commutative ring R and a group G , if M is an RG -bimodule, then it may be regarded as a left RG -module via the diagonal action $g \cdot m = gm g^{-1}$, where $g \in G, m \in M$. Conversely, any left RG -module M may be considered as an RG -bimodule by letting RG act trivially on the right. Together with the Eckmann-Shapiro Lemma, this shows that the Hochschild cohomology ring, $\mathrm{HH}^*(RG, M) := \bigoplus_{n \geq 0} \mathrm{Ext}_{RG \otimes_R RG^{op}}^n(RG, M)$ with coefficients in an RG -bimodule M , is the same as the usual group cohomology ring, $H^*(G, M) := \bigoplus_{n \geq 0} \mathrm{Ext}_{RG}^n(R, M)$ with coefficients in M under the diagonal action. In particular, by considering RG as its own bimodule, $\mathrm{HH}^*(RG, RG)$ is isomorphic to $H^*(G, RG)$, where RG is a left RG -module via conjugation. From this identification and the Eckmann-Shapiro Lemma, one can prove that $\mathrm{HH}^*(RG, RG)$ may be decomposed as a direct sum of the cohomology of the centralizers of conjugacy class representatives of G ([6], Theorem 2.11.2). In 1999, Siegel and Witherspoon then described a formula for the products in $\mathrm{HH}^*(RG, RG)$ in terms of this additive decomposition [11]. When G is abelian, Cibils and Solotar proved that the Hochschild cohomology ring of G is (isomorphic to) the tensor product over R of RG and its usual cohomology ring [9].

In 1952, John Tate introduced a group cohomology theory that is based on complete resolutions, and hence, expands the study of group cohomology to negative degrees [12]. Tate cohomology exploits the similarities between the usual homology and cohomology of a finite group G . A lot of study about this new cohomology has been done and can be found in ([7], Ch. VI) or ([8], Ch. XII). In particular, in 1992, Benson and Carlson proved that for a finite group G and a field \mathbf{k} of characteristic $p > 0$, if the depth of $H^*(G, \mathbf{k})$ is greater than one, products of elements in negative cohomology are zero [2]. More recently, Bergh and Jorgensen combined the

notions of Hochschild cohomology and Tate cohomology to form the Tate-Hochschild cohomology [3]. In this paper, we explore the structure of the Tate-Hochschild cohomology of a group algebra and generalize some known results about group cohomology to its negative degrees.

The material is organized as follows. In Section 2, we recall the definition and properties of Tate cohomology. Let M be a left RG -module. Then for all $n \in \mathbb{Z}$:

$$\widehat{H}^n(G, M) := \widehat{\text{Ext}}_{RG}^n(R, M).$$

The readers who are familiar with Tate cohomology can skip this section. In Section 3, we specialize the definition of Tate-Hochschild cohomology in [3] to the group algebra. Let M be an RG -bimodule. Then for all $n \in \mathbb{Z}$:

$$\widehat{HH}^n(RG, M) := \widehat{\text{Ext}}_{RG \otimes_R RG^{op}}^n(RG, M).$$

In this section, we also briefly present a cup product that makes the Tate-Hochschild cohomology

$$\widehat{HH}^*(RG, RG) = \bigoplus_{n \in \mathbb{Z}} \widehat{HH}^n(RG, RG)$$

become a graded ring [10]. In Section 4, we reduce the Tate-Hochschild cohomology of RG to its Tate cohomology using a similar argument as in the usual cohomology. In particular, we show that $\widehat{HH}^*(RG, RG)$ is isomorphic to $\widehat{H}^*(G, RG)$, where RG is a left RG -module via conjugation. In addition, we state known Tate cohomology relations with subgroups, which will be useful in proving our main result. In Section 5, we introduce a generalization of the Tate-Hochschild cohomology ring of RG by letting another finite group H act on G and consider $\widehat{H}^*(H, RG)$. Let $g_1, \dots, g_t \in G$ be representatives of the orbits of the action of H on G . Let $H_i := \text{Stab}_H(g_i) = \{h \in H \mid {}^h g_i = g_i\}$ be the stabilizer of g_i . We show that $\widehat{H}^*(H, RG)$ decomposes as:

$$\widehat{H}^*(H, RG) \cong \bigoplus_i \widehat{H}^*(H_i, R).$$

In the main Theorem 5.5, we describe the multiplicative structure of $\widehat{H}^*(H, RG)$ by giving it a product formula in terms of this additive decomposition. Our work, which uses mainly the idea from [11], is a straightforward generalization of the usual group cohomology results. We observe that the Tate-Hochschild cohomology ring $\widehat{HH}^*(RG, RG) \cong \widehat{H}^*(G, RG)$ is a special case of $\widehat{H}^*(H, RG)$ by letting $H = G$ act on itself by conjugation. Working with this generalization, we prove that the Tate-Hochschild cohomology of G decomposes as a direct sum of the Tate cohomology of the centralizers of conjugacy class representatives of G with coefficients in R . Moreover, the product formula reduces the computation of products in $\widehat{HH}^*(RG, RG)$ to products within the Tate cohomology rings of certain subgroups of G . When G is abelian, Cibils and Solotar's result in [9] is also generalized to the isomorphism $\widehat{HH}^*(RG, RG) \cong RG \otimes_R \widehat{H}^*(G, R)$ in Proposition 5.1. In Section 6, we describe the Tate-Hochschild cohomology ring of the symmetric group on three elements S_3 over a field \mathbf{k} of characteristic 3, utilizing the product formula in Theorem 5.5.

2. TATE COHOMOLOGY OF A GROUP ALGEBRA

Throughout this paper, we let G be a finite group and R be the ring of integers \mathbb{Z} or a field \mathbf{k} of characteristic $p > 0$ such that p divides the order of G . All rings and algebras are assumed to possess a unit; all modules are assumed to be left modules unless stated otherwise; and

tensor products will be over R unless stated otherwise. If $R = \mathbf{k}$ is a field, then $\mathbf{k}G$ is a finite dimensional, symmetric, and self-injective algebra over \mathbf{k} ([5], Prop. 3.1.2), and hence, projective $\mathbf{k}G$ -modules are the same as injective $\mathbf{k}G$ -modules. If G is acting on a set X , then we denote the action ${}^g x = gxg^{-1}$, for all $g \in G$ and $x \in X$.

Let M and N be left RG -modules. Then for any $g \in G$, $m \in M$, $n \in N$, and $f \in \text{Hom}_R(M, N)$, we observe some basic facts:

- $M \otimes N$ is a left RG -module via $g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n)$,
- $\text{Hom}_R(M, N)$ is a left RG -module via $(g \cdot f)(m) = g(f(g^{-1} \cdot m))$,
- We may regard left RG -modules as right RG -modules via $m \cdot g = g^{-1} \cdot m$.

The Tate cohomology for RG is defined in both positive and negative degrees using the following general resolution:

Definition 2.1. *Let \mathcal{R} be a two-sided Noetherian ring. A **complete resolution** of a finitely generated \mathcal{R} -module M is an exact complex $\mathbb{P} = \{\{P_i\}_{i \in \mathbb{Z}}, d_i : P_i \rightarrow P_{i-1}\}$ of finitely generated projective \mathcal{R} -modules such that:*

- (1) *The dual complex $\text{Hom}_{\mathcal{R}}(\mathbb{P}, \mathcal{R})$ is also exact*
- (2) *There exists a projective resolution $\mathbb{Q} \xrightarrow{\varepsilon} M$ of M and a chain map $\mathbb{P} \xrightarrow{\varphi} \mathbb{Q}$ where φ_n is bijective for $n \geq 0$ and 0 for $n < 0$.*

We explicitly construct an RG -complete resolution of R as follows. Let

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} R \rightarrow 0$$

be an RG -projective resolution of R , where each P_i is a finitely generated projective RG -module. Apply $\text{Hom}_R(-, R)$ to get a dual sequence:

$$0 \rightarrow R \rightarrow \text{Hom}_R(P_0, R) \rightarrow \text{Hom}_R(P_1, R) \rightarrow \text{Hom}_R(P_2, R) \rightarrow \cdots,$$

which is an exact sequence of RG -modules, since $R \cong \text{Hom}_R(R, R)$ and $\text{Hom}_R(-, R)$ is an exact functor when $R = \mathbf{k}$ is a field (the case $R = \mathbb{Z}$ is shown in [8], Prop. XII.3.3). Splicing these two sequences together, one forms a doubly infinite sequence:

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow \text{Hom}_R(P_0, R) \rightarrow \text{Hom}_R(P_1, R) \rightarrow \cdots$$

We use the notation $P_{-(n+1)} := \text{Hom}_R(P_n, R)$. By definition, the above sequence is an RG -complete resolution of R :

$$\mathbb{P} : \quad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$$

Let M be any (left) RG -module. Apply $\text{Hom}_{RG}(-, M)$ to \mathbb{P} and take the homology of this new complex, we obtain the **Tate cohomology** for RG :

$$\widehat{H}^n(G, M) := \widehat{\text{Ext}}_{RG}^n(R, M) = H^n(\text{Hom}_{RG}(\mathbb{P}, M)), \text{ for all } n \in \mathbb{Z}.$$

Observe that in our context, naturally, the Tate (co)homology is independent of the complete resolution of R ([1], Theorem 5.2 and Lemma 5.3). One can see this by applying a complete chain map between two complete resolutions of R in both positive and negative degrees. This is the generalized Comparison Theorem on complete resolutions ([7], Prop. VI.3.3).

We note that the Tate cohomology of RG obtains a multiplicative structure as described in ([7], Section VI.5) and ([8], Sections XII.4 and XII.5). In particular, if M is a ring on which G

acts by automorphisms: $g \cdot (m_1 m_2) = (g \cdot m_1)(g \cdot m_2)$, then $\widehat{H}^*(G, M)$ becomes an associative graded ring. It is graded-commutative, in the sense that, for $\alpha \in \widehat{H}^i(G, M)$ and $\beta \in \widehat{H}^j(G, M)$, $\alpha\beta = (-1)^{ij}\beta\alpha$ ([8], Props. XII.5.2 and XII.5.3). When $M = R$, we denote $\widehat{H}^*(G) := \widehat{H}^*(G, R)$.

Moreover, for a left RG -module M , we recall the following properties of Tate cohomology ([7], Section VI.5):

- (a) For all $n > 0$, $\widehat{H}^n(G, M) \cong H^n(G, M)$.

This follows from the construction of a complete resolution of R . The positive-degree component of a complete resolution of R arises from a projective resolution of R which is also used to form the usual cohomology groups $H^n(G, M)$.

- (b) The group $\widehat{H}^0(G, M)$ is a quotient of $H^0(G, M)$.

- (c) The group $\widehat{H}^{-1}(G, M)$ is the dually defined submodule of $H_0(G, M)$.

These follow from the construction of complete resolutions.

- (d) For all $n < -1$, we have isomorphisms: $\widehat{H}^n(G, M) \cong H_{-(n+1)}(G, M)$.

For any left finitely generated RG -module P_t , the map $\tau : \text{Hom}_R(P_t, R) \otimes_{RG} M \rightarrow \text{Hom}_{RG}(P_t, M)$ is an RG -module isomorphism ([8], Section XII.3 (5)). It is defined explicitly by: $\tau(f \otimes_{RG} m)(p) = \sum_{g \in G} f(g^{-1}p)gm$, where $m \in M, p \in P_t$, and $f \in \text{Hom}_R(P_t, R)$. With $t = n < -1$, $P_n := \text{Hom}_R(P_{-(n+1)}, R)$. Since $P_{-(n+1)}$ is finitely generated, $\text{Hom}(P_n, R) = \text{Hom}(\text{Hom}(P_{-(n+1)}, R), R) \cong P_{-(n+1)}$. Hence, by the duality isomorphism, we have $P_{-(n+1)} \otimes_{RG} M \cong \text{Hom}_{RG}(P_n, M)$, which induces the desired isomorphism on homology.

- (e) If $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ is a short exact sequence of (left) RG -modules, then there is a doubly infinite exact sequence of Tate cohomology groups:

$$\cdots \rightarrow \widehat{H}^n(G, M) \rightarrow \widehat{H}^n(G, M') \rightarrow \widehat{H}^n(G, M'') \rightarrow \widehat{H}^{n+1}(G, M) \rightarrow \cdots$$

This is true since the spliced complex \mathbb{P} consists of projective modules, so the short exact sequence of modules $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ gives rise to a short exact sequence of complexes:

$$0 \rightarrow \text{Hom}_{RG}(\mathbb{P}, M) \rightarrow \text{Hom}_{RG}(\mathbb{P}, M') \rightarrow \text{Hom}_{RG}(\mathbb{P}, M'') \rightarrow 0$$

whose cohomology long exact sequence is the desired long exact sequence ([1], Prop. 5.4).

- (f) If $(N_j)_{j \in J}$ is a finite family of (left) RG -modules and $(M_i)_{i \in I}$ is any family of RG -modules, then there are natural isomorphisms, for all $n \in \mathbb{Z}$:

$$\begin{aligned} \widehat{\text{Ext}}_{RG}^n(\bigoplus_{j \in J} N_j, M) &\cong \prod_{j \in J} \widehat{\text{Ext}}_{RG}^n(N_j, M) \\ \widehat{\text{Ext}}_{RG}^n(N, \prod_{i \in I} M_i) &\cong \prod_{i \in I} \widehat{\text{Ext}}_{RG}^n(N, M_i). \end{aligned}$$

The idea of this proof is similar to that of the usual Ext_{RG}^n , using the analogous relation for Hom ([1], Prop. 5.7).

In 2011, Bergh and Jorgensen introduced the notion of Tate-Hochschild cohomology which is the extended Hochschild cohomology using complete resolutions [3]. One natural question

to ask is whether the Tate-Hochschild cohomology shares the same properties as those of the usual Hochschild cohomology. In the following sections, we will examine the Tate-Hochschild cohomology of RG and its ring structure.

3. TATE-HOCHSCHILD COHOMOLOGY OF A GROUP ALGEBRA

The opposite algebra of RG , denoted by RG^{op} , is RG with the same addition but with multiplication performed in the reverse order. Let $RG^e = RG \otimes RG^{op}$ be the enveloping algebra of RG . Since G is finite, RG^e is two-sided Noetherian and Gorenstein of Gorenstein dimension 0. Let M be an RG -bimodule. M may be regarded as a left RG^e -module by setting $(a \otimes b) \cdot m = amb$, where $a \in RG, b \in RG^{op}$, and $m \in M$. In particular, we shall regard RG as a left RG^e -module. For any integer $n \in \mathbb{Z}$, the n -th Tate-Hochschild cohomology of RG is defined as:

$$\widehat{HH}^n(RG, M) := \widehat{\text{Ext}}_{RG^e}^n(RG, M),$$

where the $\widehat{\text{Ext}}$ functor is taken using an RG^e -complete resolution of RG . As RG^e is a two-sided Noetherian and Gorenstein ring, Theorems 3.1 and 3.2 in [1] guarantee that every finitely generated RG^e -module admits a complete resolution. Hence, we obtain an RG^e -complete resolution \mathbb{X} for RG . The n -th Tate-Hochschild cohomology group of RG is the n -th homology group of the complex $\text{Hom}_{RG^e}(\mathbb{X}, M)$.

Remark. The Tate-Hochschild cohomology groups of RG agree with the usual Hochschild cohomology groups in all positive degrees:

$$\widehat{HH}^n(RG, M) \cong HH^n(RG, M), \text{ for all } n > 0.$$

Let M and N be RG -bimodules. Then $M \otimes_{RG} N$ is also an RG -bimodule, which can be considered as a left RG^e -module via $(a \otimes b) \cdot (m \otimes_{RG} n) = am \otimes_{RG} nb$, where $a \in RG, b \in RG^{op}, m \in M$, and $n \in N$. There is a cup product on Tate-Hochschild cohomology:

$$\widehat{HH}^i(RG, M) \otimes \widehat{HH}^j(RG, N) \rightarrow \widehat{HH}^{i+j}(RG, M \otimes_{RG} N),$$

which we describe as follows. Let \mathbb{X} be any RG^e -complete resolution of RG . The complete tensor product $\mathbb{X} \widehat{\otimes}_{RG} \mathbb{X}$ is obtained by defining:

$$(\mathbb{X} \widehat{\otimes}_{RG} \mathbb{X})_n = \prod_{i+j=n} X_i \otimes_{RG} X_j, \text{ for all } n \in \mathbb{Z}.$$

By the same argument as in ([7], Section VI.5) or ([10], Section 6.1), $\mathbb{X} \widehat{\otimes}_{RG} \mathbb{X}$ is an acyclic chain complex of RG^e -modules. Lemma 6.5 in [10] shows the existence of a complete diagonal approximation chain map $\Gamma : \mathbb{X} \rightarrow \mathbb{X} \widehat{\otimes}_{RG} \mathbb{X}$ that preserves the augmentation.

Given graded modules B, B', C, C' and module homomorphisms $u : C \rightarrow B$ of degree i and $v : C' \rightarrow B'$ of degree j , there is a map $u \widehat{\otimes} v : C \widehat{\otimes} C' \rightarrow B \widehat{\otimes} B'$ of degree $i + j$ defined by:

$$(u \widehat{\otimes} v)_n = \prod_{r+s=n} (-1)^{rj} u_r \otimes v_s : \prod_{r+s=n} C_r \otimes C'_s \rightarrow \prod_{r+s=n} B_{r+i} \otimes B'_{s+j}.$$

Let $f \in \text{Hom}_{RG^e}(X_i, M)$ represent an element of $\widehat{HH}^i(RG, M)$ and let $g \in \text{Hom}_{RG^e}(X_j, N)$ represent an element of $\widehat{HH}^j(RG, N)$. Then:

$$f \smile g = (f \widehat{\otimes} g) \circ \Gamma \in \text{Hom}_{RG^e}(\mathbb{X}, M \otimes_{RG} N)$$

represents an element of $\widehat{\mathrm{HH}}^{i+j}(RG, M \otimes_{RG} N)$. One can check that this product is independent of \mathbb{X} and Γ and satisfies the usual cup product properties. When $M = N = RG$, this cup product gives $\widehat{\mathrm{HH}}^*(RG, RG)$ the structure of an associative graded ring, as $RG \cong RG \otimes_{RG} RG$.

4. REDUCTION TO TATE COHOMOLOGY AND RELATIONS WITH SUBGROUPS

We show here that the Tate-Hochschild cohomology of RG can be reduced to its Tate cohomology. We begin with a lemma which is based on the original Eckmann-Shapiro Lemma but is generalized to a complete resolution:

Lemma 4.1 (Eckmann-Shapiro, ([10], Lemma 7.1) or ([7], VI.5.2)). *Let H be a subgroup of a finite group G , M be a left RH -module and N be a left RG -module. Consider $N \downarrow_H^G = N$ to be a left RH -module via restriction of the action, and let $M \uparrow_H^G := RG \otimes_{RH} M$ denote the induced RG -module where G acts on the leftmost factor by multiplication. Then for all $n \in \mathbb{Z}$, there is an isomorphism of abelian groups:*

$$\widehat{\mathrm{Ext}}_{RH}^n(M, N \downarrow_H^G) \cong \widehat{\mathrm{Ext}}_{RG}^n(M \uparrow_H^G, N).$$

For any subgroup $H \subseteq G$, every RH -complete resolution gives rise to an RG -complete resolution by inducing the modules (induction takes projectives to projectives, and exact sequences to exact sequences). Since coinduction is the same as induction in the finite group case ([7], Prop. III.5.9), the proof for this lemma goes through in the present context and follows from the Nakayama relations ([5], Prop. 2.8.3).

We have $RG \cong RG^{op}$ as algebras via $g \mapsto g^{-1}$, and $RG \otimes RG \cong R(G \times G)$ as algebras. As a result, $RG^e = RG \otimes RG^{op} \cong R(G \times G)$. Here, RG is a left $R(G \times G)$ -module by the two-sided action: $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. So RG is just the permutation module $R \uparrow_{\delta(G)}^{G \times G} = R(G \times G) \otimes_{R\delta(G)} R$ on the cosets of the diagonal $\delta(G) = \{(g, g) \mid g \in G\}$. Moreover, any RG -bimodule M may be considered as a left RG -module with action given by $g \cdot m = g m g^{-1}$, for all $g \in G$ and $m \in M$. It follows from this discussion and Lemma 4.1 that:

$$\begin{aligned} \widehat{\mathrm{HH}}^*(RG, M) &= \widehat{\mathrm{Ext}}_{RG^e}^*(RG, M) \cong \widehat{\mathrm{Ext}}_{R(G \times G)}^*(RG, M) \\ &\cong \widehat{\mathrm{Ext}}_{R(G \times G)}^*(R \uparrow_{\delta(G)}^{G \times G}, M) \\ &\cong \widehat{\mathrm{Ext}}_{RG}^*(R, M \downarrow_{\delta(G)}^{G \times G}) \\ &\cong \widehat{H}^*(G, M). \end{aligned}$$

The Tate-Hochschild cohomology of G with coefficients in the bimodule M is just the Tate cohomology of G with coefficients in M under the diagonal action. When M is an algebra itself with compatible G -action, this is in fact an algebra isomorphism as it respects the cup products defined for Tate and Tate-Hochschild cohomology ([10], Theorem 7.2). In particular, when $M = RG$ with G -action via conjugation,

$$\widehat{\mathrm{HH}}^*(RG, RG) \cong \widehat{H}^*(G, RG). \quad (4.2)$$

Therefore, all properties of $\widehat{H}^*(G, RG)$ that we observed in Section 2 transfer to those for the Tate-Hochschild cohomology.

Let H be a subgroup of G . By restricting the action, any RG -module N may be regarded as an RH -module and any RG -complete resolution \mathbb{P} of R may also be considered as an RH -complete

resolution. Sections XII.8 and XII.9 in [8] show that there are maps in the Tate cohomology with properties analogous to those in the usual group cohomology:

- The restriction map:

$$\text{res}_H^G : \widehat{H}^*(G, N) \rightarrow \widehat{H}^*(H, N),$$

which is induced from the inclusion $\text{Hom}_{RG}(\mathbb{P}, N) \subset \text{Hom}_{RH}(\mathbb{P}, N)$.

- The corestriction map (or transfer):

$$\text{cor}_H^G : \widehat{H}^*(H, N) \rightarrow \widehat{H}^*(G, N),$$

which is given on the cochain level by defining:

$$(\text{cor}_H^G f)(p) = \sum_{g \in \mathcal{G}} gf(g^{-1}p),$$

where \mathcal{G} denotes a set of left coset representatives of H in G , $f \in \text{Hom}_{RH}(P_i, N)$, and $p \in P_i$. One can check that this definition is independent of the choice of the representatives $g \in \mathcal{G}$.

- Moreover, for any $g \in G$, there is an isomorphism:

$$g^* : \widehat{H}^*(H, N) \rightarrow \widehat{H}^*(gHg^{-1}, N) = \widehat{H}^*({}^gH, N)$$

defined on the cochain level as $(g^*f)(p) = g(f(g^{-1}p))$.

We shall list some properties of these maps without proving them. The proof goes through using similar arguments as in [8]. The readers can refer to [8] for more details.

Proposition 4.3 ([8], Section XII.8 (4)-(14) and Section XII.9 (4)). *Let $K \subseteq H \subseteq G$ be subgroups, and N_1, N_2 be RG -modules which may be regarded as RH -modules. Let $\alpha_i \in \widehat{H}^*(G, N_i)$, $\beta_i \in \widehat{H}^*(H, N_i)$, and $g_i \in G$, for $i = 1, 2$. Then the maps defined above satisfy:*

- (1) $g_1^*g_2^* = (g_1g_2)^*$
- (2) $g^* = \mathbf{1}$, if $g \in H$
- (3) $\text{cor}_H^G \circ \text{res}_H^G = (G : H)\mathbf{1}$
- (4) $\text{res}_K^H \circ \text{res}_H^G = \text{res}_K^G$
- (5) $\text{cor}_H^G \circ \text{cor}_K^H = \text{cor}_K^G$
- (6) $g^* \circ \text{res}_K^H = \text{res}_{gK}^{gH} \circ g^*$
- (7) $g^* \circ \text{cor}_K^H = \text{cor}_{gK}^{gH} \circ g^*$
- (8) $\text{res}_H^G(\alpha_1 \smile \alpha_2) = (\text{res}_H^G \alpha_1) \smile (\text{res}_H^G \alpha_2)$
- (9) $\text{cor}_H^G(\beta_1 \smile \text{res}_H^G \alpha_2) = (\text{cor}_H^G \beta_1) \smile \alpha_2$
- (10) $\text{cor}_H^G(\text{res}_H^G \alpha_1 \smile \beta_2) = \alpha_1 \smile (\text{cor}_H^G \beta_2)$
- (11) $g^*(\beta_1 \smile \beta_2) = (g^*\beta_1) \smile (g^*\beta_2)$
- (12) *Let $H, K \subseteq G$ be subgroups and N be an RG -module which may be regarded as an RH (or RK)-module. The map $\text{res}_K^G \circ \text{cor}_H^G : \widehat{H}^*(H, N) \rightarrow \widehat{H}^*(K, N)$ is given by:*

$$\text{res}_K^G(\text{cor}_H^G(\beta)) = \sum_{x \in D} \text{cor}_{K \cap {}^xH}^K(\text{res}_{K \cap {}^xH}^{xH}(x^*\beta)),$$

where $\beta \in \widehat{H}^*(H, N)$ and D is a set of double coset representatives such that $G = \bigcup_{x \in D} KxH$ is a disjoint union.

5. GENERALIZED ADDITIVE DECOMPOSITION

For the rest of this paper, we let H be another finite group which acts as automorphisms on G . Via this action, RG becomes an RH -module. The multiplication map $RG \otimes RG \rightarrow RG$ is an RH -module homomorphism. Hence, it induces the ring structure on cohomology $\widehat{H}^*(H, RG) := \widehat{\text{Ext}}_{RH}^*(R, RG)$ by compositing with the cup product. We will study the additive decomposition of this ring $\widehat{H}^*(H, RG)$. The Tate-Hochschild cohomology ring $\widehat{H}^*(RG, RG) \cong \widehat{H}^*(G, RG)$ is a special case of $\widehat{H}^*(H, RG)$ by letting $H = G$ act on itself by conjugation.

Proposition 5.1. *If H acts trivially on G , then $\widehat{H}^*(H, RG) \cong RG \otimes_R \widehat{H}^*(H, R)$ as graded R -algebras. In particular, if G is abelian, then*

$$\widehat{HH}^*(RG, RG) \cong RG \otimes_R \widehat{H}^*(G, R).$$

Proof. If G is abelian, $H = G$ acting on itself by conjugation yields the trivial action. Hence, the second statement follows from the first statement and the isomorphism (4.2):

$$\widehat{HH}^*(RG, RG) \cong \widehat{H}^*(G, RG) \cong RG \otimes_R \widehat{H}^*(G, R).$$

To prove the first statement, let $\varepsilon : \mathbb{P} \rightarrow R$ be an RH -complete resolution of R . Since H acts trivially on G , RG is a trivial RH -module and is free as an R -module. We claim that $\gamma : RG \otimes \text{Hom}_{RH}(\mathbb{P}, R) \rightarrow \text{Hom}_{RH}(\mathbb{P}, RG)$ is an isomorphism. It can be seen by sending $g \otimes f \mapsto \gamma(g \otimes f) = F$, where $F(p) = f(p)g$, for $f \in \text{Hom}_{RH}(P_i, R)$, $p \in P_i$, and $g \in G$. It is easy to check that $F \in \text{Hom}_{RH}(\mathbb{P}, RG)$ and it is a cocycle when f is. Hence, passing to the homology, γ induces an isomorphism of graded R -modules:

$$\gamma_* : RG \otimes_R \widehat{H}^*(H, R) \rightarrow \widehat{H}^*(H, RG).$$

The definition of cup product corresponds to this map, making γ_* a ring homomorphism. \square

This proposition helps us to find and study the structure of the Tate-Hochschild cohomology ring of a finite abelian group algebra, given its Tate cohomology ring. For example, knowing the Tate cohomology of a cyclic group G , (see [8], Section XII.7), one can easily compute its Tate-Hochschild cohomology by applying Proposition 5.1.

Now we return to the general case where H acts on G non-trivially and G is not necessarily abelian. Let $g_1, \dots, g_t \in G$ be representatives of the orbits of the action of H on G . Let $H_i := \text{Stab}_H(g_i) = \{h \in H \mid {}^h g_i = g_i\}$ be the stabilizer of g_i . For any $g \in G$, there are two $R(\text{Stab}_H(g))$ -module homomorphisms:

$$\begin{aligned} \theta_g : R &\rightarrow RG \text{ via } r \mapsto rg, \\ \pi_g : RG &\rightarrow R \text{ via } \sum_{a \in G} r_a a \mapsto r_g. \end{aligned}$$

If V is any subgroup of $\text{Stab}_H(g)$, then these maps induce maps on cohomology:

$$\begin{aligned} \theta_g^* : \widehat{H}^*(V, R) &\rightarrow \widehat{H}^*(V, RG), \\ \pi_g^* : \widehat{H}^*(V, RG) &\rightarrow \widehat{H}^*(V, R), \end{aligned}$$

since $\widehat{\text{Ext}}^*$ is covariant in the second argument. The following properties of θ_g^* and π_g^* will help us in proving the main result.

Lemma 5.2. *Let $h \in H$ and $a, b \in G$.*

- (a) If V is a subgroup of $\text{Stab}_H(a)$, then $h^* \circ \theta_a^* = \theta_{h_a}^* \circ h^*$ as maps from $\widehat{H}^*(V)$ to $\widehat{H}^*({}^hV, RG)$.
 (b) Suppose $V \subseteq \text{Stab}_H(a) \cap \text{Stab}_H(b)$ and $\alpha, \beta \in \widehat{H}^*(V)$. Then:

$$\theta_a^*(\alpha) \smile \theta_b^*(\beta) = \theta_{ab}^*(\alpha \smile \beta).$$

- (c) Suppose $V' \subseteq V \subseteq \text{Stab}_H(a)$. Then θ_a^* and π_a^* commute with $\text{res}_{V'}^V$ and $\text{cor}_{V'}^V$.
 (d) If $V \subseteq \text{Stab}_H(a) \cap \text{Stab}_H(b)$, then $\pi_a^* \circ \theta_b^* = \delta_{a,b} \mathbf{1}$, where $\mathbf{1}$ is the identity map on $\widehat{H}^*(V)$ and $\delta_{a,b}$ is the Kronecker delta.

Proof. Lemma 5.2 in [11] showed these properties in positive degrees. We generalize the proof to negative degrees and present it on the cochain level. The desired results are induced on cohomology.

- (a) Let \mathbb{P} be an RV -complete resolution of R , $f \in \text{Hom}_V(P_i, R)$ be a cocycle representing an element of $\widehat{H}^i(V)$, and $p \in P_i$. Then

$$h^*(\theta_a f)(p) = f(h^{-1}p)({}^h a) = \theta_{h_a}(h^*(f))(p).$$

- (b) Let $m : RG \otimes RG \rightarrow RG$ be the multiplication map and $\Gamma : \mathbb{P} \rightarrow \mathbb{P} \widehat{\otimes} \mathbb{P}$ be a complete diagonal approximation map. Let $f, q \in \text{Hom}_V(\mathbb{P}, R)$ represent $\alpha, \beta \in \widehat{H}^*(V)$, respectively. Then on the cochain level:

$$m \circ ((\theta_a \circ f) \widehat{\otimes} (\theta_b \circ q)) \circ \Gamma = m \circ (\theta_a \otimes \theta_b) \circ (f \widehat{\otimes} q) \circ \Gamma = \theta_{ab} \circ (f \widehat{\otimes} q) \circ \Gamma,$$

where the left side represents $\theta_a^*(\alpha) \smile \theta_b^*(\beta)$ and the right side represents $\theta_{ab}^*(\alpha \smile \beta)$.

- (c) Let \mathbb{P} be an RV -complete resolution of R which can also be regarded as an RV' -complete resolution by restricting the action. Let $f \in \text{Hom}_{V'}(P_i, RG)$ represent an element of $\widehat{H}^i(V', RG)$, $q \in \text{Hom}_{V'}(P_i, R)$ represent an element of $\widehat{H}^i(V')$, and $p \in P_i$.

$$\begin{aligned} (\pi_a^* \text{cor}_{V'}^V)(f)(p) &= \pi_a \left(\sum_{v \in V/V'} ({}^v f(v^{-1}p)) \right) = \sum_v \pi_{v^{-1}a}(f(v^{-1}p)) \\ &= \sum_v (\pi_a \circ f)(v^{-1}p) = (\text{cor}_{V'}^V \pi_a^*)(f)(p), \end{aligned}$$

since $v \in V \subseteq \text{Stab}_H(a)$, we have $v^{-1}a = a$, and V acts trivially on R . Similarly,

$$\begin{aligned} (\theta_a^* \text{cor}_{V'}^V)(q)(p) &= \theta_a \left(\sum_{v \in V/V'} q(v^{-1}p) \right) = \sum_v \theta_{v^{-1}a}(q(v^{-1}p)) \\ &= \sum_v (\theta_a \circ q)(v^{-1}p) = (\text{cor}_{V'}^V \theta_a^*)(q)(p). \end{aligned}$$

The other cases follow similarly by commutativity between π_a, θ_a and the inclusion map $\iota : \text{Hom}_V(\mathbb{P}, N) \hookrightarrow \text{Hom}_{V'}(\mathbb{P}, N)$, where $N = RG$ or R .

- (d) Let $r \in R$.

$$\pi_a(\theta_b(r)) = \pi_a(rb) = \begin{cases} r, & \text{if } a = b \\ 0, & \text{else.} \end{cases}$$

□

For all $i = 1, 2, \dots, t$, let $\psi_i : \widehat{H}^*(H_i, R) \rightarrow \widehat{H}^*(H, RG)$ be defined as the composition $\psi_i = \text{cor}_{H_i}^H \circ \theta_{g_i}^*$. We describe the additive decomposition that is generalized from the usual cohomology ([6], Theorem 2.11.2):

Lemma 5.3. *The map $\widehat{H}^*(H, RG) \rightarrow \bigoplus_i \widehat{H}^*(H_i, R)$, sending $\zeta \mapsto (\pi_{g_i}^* \circ \text{res}_{H_i}^H(\zeta))_i$, is an isomorphism of graded R -modules, for $\zeta \in \widehat{H}^*(H, RG)$. Its inverse sends $\alpha \in \widehat{H}^*(H_i, R)$ to $\psi_i(\alpha) \in \widehat{H}^*(H, RG)$.*

Proof. For $i = 1, 2, \dots, t$, let M_i be the free R -module generated by elements of the orbit containing g_i . Then $RG = \bigoplus_i M_i$. There is an isomorphism $M_i \rightarrow R \uparrow_{H_i}^H = RH \otimes_{RH_i} R$ given by $r({}^h g_i) \mapsto h \otimes r$. It induces an isomorphism in cohomology $\widehat{H}^*(H, M_i) \cong \widehat{H}^*(H, R \uparrow_{H_i}^H)$.

Since $\widehat{\text{Ext}}$ is additive, $\widehat{H}^*(H, RG) \cong \bigoplus_i \widehat{H}^*(H, M_i)$. Apply the Eckmann-Shapiro Lemma 4.1, we have $\widehat{H}^*(H, RG) \cong \bigoplus_i \widehat{H}^*(H_i, R)$. One can also check directly that the maps given in the statement of the lemma are inverses of each other by taking their compositions and applying Proposition 4.3 and Lemma 5.2 to show that their compositions are the identity maps. \square

Remark. If $H = G$ acts on itself by conjugation, then M_i is the free R -module generated by the conjugacy class of g_i . M_i is isomorphic to $R \uparrow_{C_G(g_i)}^G$, where $C_G(g_i)$ is the centralizer of g_i . Therefore, the isomorphism in Lemma 5.3 gives an additive decomposition of the Tate-Hochschild cohomology of G as a direct sum of the Tate cohomology of the centralizers of conjugacy class representatives of G with coefficients in R :

$$\widehat{HH}^*(RG, RG) \cong \bigoplus_i \widehat{H}^*(C_G(g_i), R).$$

In 1999, Siegel and Witherspoon showed that there is a product formula for the usual Hochschild cohomology of G in terms of a similar additive decomposition ([11], Theorem 5.1). We will describe products in $\widehat{H}^*(H, RG)$ with respect to the isomorphism in Lemma 5.3. The argument will be analogous to that in [11] but is generalized to the Tate cohomology.

Fix $i, j \in \{1, 2, \dots, t\}$. Let D be a set of double coset representatives for $H_i \backslash H / H_j$. For each $x \in D$, there is a unique $k = k(x)$ such that

$$g_k = {}^y g_i {}^{yx} g_j \quad (5.4)$$

for some $y \in H$. One can expand the action on the right hand side and get $g_k = {}^y(g_i {}^x g_j)$ showing that g_k is just a representative of an orbit of the action of H on the double coset $H_i x H_j$. Moreover, k is independent of the choice of double coset representative x . The set of all y satisfying (5.4) is also a double coset. To see this, let us fix $y = y(x)$ for which (5.4) holds. Let $y' \in H$ be another element such that $g_k = {}^{y'} g_i {}^{y'x} g_j$. Then ${}^{y'} g_i {}^{y'x} g_j = g_k = {}^y g_i {}^{yx} g_j$ implies ${}^{y'}(g_i {}^x g_j) = {}^y(g_i {}^x g_j)$. Let $h = y' y^{-1}$. We have ${}^h g_k = {}^h(y(g_i {}^x g_j)) = {}^{y'}(g_i {}^x g_j) = g_k$ showing $h \in H_k = \text{Stab}_H(g_k)$. On the other hand, if $h \in H_k$, then let $y' = hy \in H$, we have ${}^{hy} g_i {}^{hyx} g_j = {}^h(y g_i {}^{yx} g_j) = {}^h g_k = g_k$. Putting together, we have shown:

$$\{y' \in H \mid g_k = {}^{y'} g_i {}^{y'x} g_j\} = H_k y = H_k y({}^x H_j \cap H_i) \in H_k \backslash H / ({}^x H_j \cap H_i),$$

where the last equality follows from (5.4) and ${}^{yx} H_j \cap {}^y H_i \subseteq H_k$. We can now prove our main result which provides a formula for products in $\widehat{H}^*(H, RG)$ with respect to Lemma 5.3.

Theorem 5.5. *Let $\alpha \in \widehat{H}^*(H_i)$ and $\beta \in \widehat{H}^*(H_j)$. Then*

$$\psi_i(\alpha) \smile \psi_j(\beta) = \sum_{x \in D} \psi_k(\text{cor}_V^{H_k}(\text{res}_V^{yH_i} y^* \alpha \smile \text{res}_V^{yxH_j} (yx)^* \beta))$$

where D is a set of double coset representatives for $H_i \backslash H / H_j$, $k = k(x)$ and $y = y(x)$ are chosen to satisfy (5.4), and $V = V(x) = {}^{yx} H_j \cap {}^y H_i \subseteq H_k$.

Proof. By Lemma 5.3,

$$\begin{aligned}
\psi_i(\alpha) \smile \psi_j(\beta) &= \text{cor}_{H_i}^H(\theta_{g_i}^* \alpha) \smile \text{cor}_{H_j}^H(\theta_{g_j}^* \beta), \text{ by definition of } \psi_i, \psi_j \\
&= \text{cor}_{H_i}^H(\theta_{g_i}^* \alpha \smile \text{res}_{H_i}^H \text{cor}_{H_j}^H \theta_{g_j}^* \beta), \text{ by Prop. 4.3 (9)} \\
&= \sum_{x \in D} \text{cor}_{H_i}^H(\theta_{g_i}^* \alpha \smile \text{cor}_{xH_j \cap H_i}^{H_i} \text{res}_{xH_j \cap H_i}^{xH_j} x^* \theta_{g_j}^* \beta), \text{ by Prop. 4.3 (12)} \\
&= \sum_{x \in D} \text{cor}_{H_i}^H(\text{cor}_{xH_j \cap H_i}^{H_i} (\text{res}_{xH_j \cap H_i}^{H_i} \theta_{g_i}^* \alpha \smile \text{res}_{xH_j \cap H_i}^{xH_j} x^* \theta_{g_j}^* \beta)), \text{ by Prop. 4.3 (10)} \\
&= \sum_{x \in D} \text{cor}_{xH_j \cap H_i}^H (\text{res}_{xH_j \cap H_i}^{H_i} \theta_{g_i}^* \alpha \smile \text{res}_{xH_j \cap H_i}^{xH_j} x^* \theta_{g_j}^* \beta), \text{ by Prop. 4.3 (5)} \\
&= \sum_{x \in D} \text{cor}_{xH_j \cap H_i}^H \theta_{g_i}^* x g_j (\text{res}_{xH_j \cap H_i}^{H_i} \alpha \smile \text{res}_{xH_j \cap H_i}^{xH_j} x^* \beta), \text{ by Lemma 5.2 (a)-(c)} \\
&= \sum_k \sum_x \psi_k \pi_{g_k}^* \text{res}_{H_k}^H (\text{cor}_{xH_j \cap H_i}^H \theta_{g_i}^* x g_j (\text{res}_{xH_j \cap H_i}^{H_i} \alpha \smile \text{res}_{xH_j \cap H_i}^{xH_j} x^* \beta)), \\
&\text{by the isomorphism in Lemma 5.3} \\
&= \sum_k \sum_{x, y} \psi_k \pi_{g_k}^* \text{cor}_{V'}^{H_k} \text{res}_{V'}^{y^x H_j \cap y^y H_i} y^* \theta_{g_i}^* x g_j (\text{res}_{xH_j \cap H_i}^{H_i} \alpha \smile \text{res}_{xH_j \cap H_i}^{xH_j} x^* \beta), \\
&\text{by Prop. 4.3 (12), where } y \text{ runs over a set of representatives for } H_k \backslash H / xH_j \cap H_i \\
&\text{and } V' = H_k \cap y^x H_j \cap y^y H_i, \\
&= \sum_k \sum_{x, y} \psi_k \text{cor}_{V'}^{H_k} \pi_{g_k}^* \theta_{y g_i}^* y^x g_j \text{res}_{V'}^{y^x H_j \cap y^y H_i} y^* (\text{res}_{xH_j \cap H_i}^{H_i} \alpha \smile \text{res}_{xH_j \cap H_i}^{xH_j} x^* \beta), \\
&\text{by Lemma 5.2 (a),(c)} \\
&= \sum_{x \in D} \psi_k \text{cor}_{V'}^{H_k} (\text{res}_{V'}^{y^y H_i} y^* \alpha \smile \text{res}_{V'}^{y^x H_j} (yx)^* \beta), \\
&\text{by Prop. 4.3 (1), (4), (6) and Lemma 5.2 (d).}
\end{aligned}$$

By Lemma 5.2 (d), the only terms that can be non-zero in the next to last step are those for which $g_k = y g_i y^x g_j$. We have seen in the discussion prior to this theorem that each x determines a unique k and double coset $H_k y(xH_j \cap H_i)$ for which this holds. Therefore, we may take $y = y(x)$ and $y^x H_j \cap y^y H_i \subseteq H_k$. Hence, $V' = V = y^x H_j \cap y^y H_i$. \square

Remark. Since the cup product is well-defined and unique ([8], Theorem XII.5.1), the sum in the statement of the theorem is independent of the choice of x and y . One can see this directly by replacing y with hy , for some $h \in H_k$. By Prop. 4.3 (6), (7), and (11), h^* respects the cup product and commutes with the restriction and corestriction maps. Moreover, since H_k acts trivially on its own cohomology, any term of the sum in the theorem is unchanged by replacing y with hy . If x is multiplied on the right by an element of H_j , the terms are unchanged for similar reasons. If x is replaced by hx , for some $h \in H_i$, then we must replace y with yh^{-1} so that (5.4) holds:

$$(yh^{-1}) g_i (yh^{-1})(hx) g_j = y g_i y^x g_j = g_k,$$

and the terms remain unchanged.

We observe that when $i = 1$, $\psi_1 : \widehat{H}^*(H, R) \rightarrow \widehat{H}^*(H, RG)$ is an algebra monomorphism that is induced by the algebra homomorphism $R \rightarrow RG$ mapping $r \mapsto r1$. Alternatively, by letting

$i = j = 1$ in Theorem 5.5, we see that ψ_1 respects the cup product:

$$\psi_1(\alpha) \smile \psi_1(\beta) = \psi_1(\alpha \smile \beta),$$

where $\alpha, \beta \in \widehat{H}^*(H)$. Hence, via ψ_1 , we may view $\widehat{H}^*(H, RG)$ as a (left) $\widehat{H}^*(H)$ -module with action via multiplying (on the left) by $\psi_1(\widehat{H}^*(H))$. Each $\widehat{H}^*(H_i)$ may also be regarded as an $\widehat{H}^*(H)$ -module via restriction. As a consequence, we obtain:

Corollary 5.6. *The isomorphism in Lemma 5.3 is an isomorphism of graded $\widehat{H}^*(H)$ -modules:*

$$\widehat{H}^*(H, RG) \xrightarrow{\cong} \bigoplus_i \widehat{H}^*(H_i)$$

Proof. For $i = 1$, let $\alpha \in \widehat{H}^*(H)$ and $\beta \in \widehat{H}^*(H_j)$. Theorem 5.5 reduces to:

$$\psi_1(\alpha) \smile \psi_j(\beta) = \psi_j(\text{res}_{H_j}^H(\alpha) \smile \beta).$$

where the left hand side is considered as action of $\widehat{H}^*(H)$ on $\widehat{H}^*(H, RG)$ that corresponds to the action of $\widehat{H}^*(H)$ on each $\widehat{H}^*(H_j)$ on the right hand side, via the isomorphism in Lemma 5.3. \square

As noted in the remark following Lemma 5.3, when $H = G$ acts on itself by conjugation, Theorem 5.5 gives a formula for the multiplicative structure of $\widehat{HH}^*(RG, RG) \cong \bigoplus_i \widehat{H}^*(C_G(g_i), R)$ in terms of this decomposition. It reduces the computation of products in $\widehat{HH}^*(RG, RG)$ to products within the Tate cohomology rings of certain subgroups of G . In the next section, we will show a basic non-abelian example that demonstrates the usefulness of Theorem 5.5.

6. THE SYMMETRIC GROUP ON THREE ELEMENTS

Let \mathbf{k} be a field of characteristic 3. Let $G = S_3 = \langle a, b \mid a^3 = 1 = b^2, ab = ba^2 \rangle$ act on itself by conjugation. Without loss of generality, we choose conjugacy class representatives $g_1 = 1$, $g_2 = a$, and $g_3 = b$ whose centralizers are $H_1 = G$, $H_2 = \{1, a, a^2\} =: N$, and $H_3 = \{1, b\}$, respectively. We will find the Tate-Hochschild cohomology ring of $\mathbf{k}G$ using elements of $\widehat{H}^*(H_i, \mathbf{k})$ and the product formula given in Theorem 5.5.

Let us examine each ring $\widehat{H}^*(H_i) := \widehat{H}^*(H_i, \mathbf{k})$. Since the characteristic of \mathbf{k} is 3 and N is cyclic of order 3, the cohomology ring $\widehat{H}^*(N)$ is periodic by ([8], Theorem XII.11.6) and ([4], (4.1.3)). By direct computation from Section XII.7 in [8], $\widehat{H}^*(N)$ is of the form $\Lambda(w_1) \otimes_{\mathbf{k}} \mathbf{k}[w_2, w_2^{-1}]$, where $\Lambda(w_1)$ is the exterior \mathbf{k} -algebra on the element w_1 of degree 1 and $\mathbf{k}[w_2, w_2^{-1}]$ is generated by the elements w_2 of degree 2 and w_2^{-1} of degree -2 , subject to the graded-commutative relations and $w_2 w_2^{-1} = 1$. By a similar computation, because the characteristic of \mathbf{k} does not divide the order of $H_3 = \{1, b\}$, we find that $\widehat{H}^n(H_3) = 0$ in all degrees. As a result, $\widehat{H}^*(H_3) = 0$.

We now compute the Tate cohomology ring of $G = S_3$ with coefficients in \mathbf{k} . N is a normal subgroup of G . The quotient group G/N is (isomorphic to) \mathbb{Z}_2 . It is easy to check that G is isomorphic to a semidirect product $N \rtimes \mathbb{Z}_2$ and every abelian subgroup of G is cyclic. It follows from ([8], Theorem XII.11.6) and ([4], (4.1.3)) that the Tate cohomology ring $\widehat{H}^*(G)$ is periodic and Noetherian. One can directly compute $\widehat{H}^*(G)$ by using an N -complete resolution of \mathbf{k} , imposing on it an action of \mathbb{Z}_2 to make it become a G -complete resolution of \mathbf{k} , computing the Tate cohomology groups from that resolution, and studying their products. Alternatively, following the discussion in ([8], Section XII.10), we see that for any G -module M , $\widehat{H}^*(G, M)$ is

a direct sum of $\widehat{H}^*(G, M, p)$, where $\widehat{H}^*(G, M, p)$ is the p -primary component of $\widehat{H}^*(G, M)$ and p runs through all the prime divisors of $|G| = 6$. Here, $M = \mathbf{k}$ is a field of characteristic 3, so only the 3-primary component is non-zero. By ([8], Theorem XII.10.1), G/N operates on $\widehat{H}^*(N)$ and so $\widehat{H}^*(G) = \widehat{H}^*(G, \mathbf{k}, 3) \cong [\widehat{H}^*(N)]^{G/N} \cong \Lambda(w_1 w_2) \otimes_{\mathbf{k}} \mathbf{k}[w_2^2, w_2^{-2}]$. Therefore, $\widehat{H}^*(G)$ is of the form $\Lambda(x) \otimes_{\mathbf{k}} \mathbf{k}[z, z^{-1}]$, where x and z are of degrees 3 and 4, respectively, subject to the graded-commutative relations.

By the decomposition Lemma 5.3, $\widehat{H}^*(G, \mathbf{k}G) \cong \widehat{H}^*(G) \oplus \widehat{H}^*(N)$ as graded \mathbf{k} -modules. We then can define elements of the Tate-Hochschild cohomology ring of $\mathbf{k}G$ as follows. Since ψ_1 is an algebra monomorphism, we may identify any element of $\widehat{H}^*(G)$ with its image under ψ_1 . Let $E_i = \psi_i(1)$, $W_i = \psi_2(w_i)$, for $i = 1, 2$, and $W_2^{-1} = \psi_2(w_2^{-1})$. For simplification, we will use $C := E_2 + 1$ in the following theorem.

Theorem 6.1. *Let \mathbf{k} be a field of characteristic 3 and S_3 be the symmetric group on three elements. Then the Tate-Hochschild cohomology $\widehat{HH}^*(\mathbf{k}S_3, \mathbf{k}S_3)$ of S_3 is generated as an algebra by elements $x, z, z^{-1}, C, W_1, W_2$, and W_2^{-1} of degrees 3, 4, -4 , 0, 1, 2, and -2 , respectively, subject to the following relations:*

$$xW_1 = 0, \quad xW_2 = zW_1, \quad z^{-1}W_1 = (xz^{-1})W_2^{-1},$$

$$C^2 = CW_2^{-1} = CW_i = 0 \quad (i = 1, 2),$$

$$W_2^2 = zC, \quad W_2^{-2} = z^{-1}C, \quad W_1W_2 = xC, \quad W_1W_2^{-1} = xz^{-1}C,$$

together with the graded-commutative relations. In particular, the algebra monomorphism $\psi_1 : \widehat{H}^*(S_3) \rightarrow \widehat{H}^*(S_3, \mathbf{k}S_3)$ induces an isomorphism modulo radicals.

Proof. $\widehat{HH}^*(\mathbf{k}G, \mathbf{k}G) \cong \widehat{H}^*(G, \mathbf{k}G)$ is a graded-commutative \mathbf{k} -algebra whose underlying \mathbf{k} -module is isomorphic to $\widehat{H}^*(G) \oplus \widehat{H}^*(N)$. Here, $\widehat{H}^*(G)$ is a graded subalgebra of $\widehat{HH}^*(\mathbf{k}G, \mathbf{k}G)$ generated by x, z and z^{-1} . Additionally, $\psi_2(\widehat{H}^*(N))$ is a graded $\widehat{H}^*(G)$ -submodule of $\widehat{HH}^*(\mathbf{k}G, \mathbf{k}G)$ generated by E_2, W_1, W_2 and W_2^{-1} . This follows from the discussion after the proof of Theorem 5.5. Moreover, we will check that these generators satisfy the following conditions:

- (1) action on $\psi_2(\widehat{H}^*(N))$ as an $\widehat{H}^*(G)$ -module, and
- (2) every product in $\psi_2(\widehat{H}^*(N))$ can be expressed as the sum of an element of $\widehat{H}^*(G)$ and a $\widehat{H}^*(G)$ -linear combination of the images under ψ_2 of the generators of $\widehat{H}^*(N)$.

Therefore, it is clear that $\widehat{HH}^*(\mathbf{k}G, \mathbf{k}G)$ is generated as a \mathbf{k} -algebra by $x, z, z^{-1}, E_2, W_1, W_2$, and W_2^{-1} , subject to these conditions. The first line of the relations in the statement of the theorem satisfies the first condition. The second and third lines satisfy the second condition. We will check each of them in detail.

The restriction $\text{res}_N^G : \widehat{H}^*(G) \rightarrow \widehat{H}^*(N)$, which sends $x \mapsto w_1 w_2$, $z \mapsto w_2^2$, and $z^{-1} \mapsto w_2^{-2}$, is injective. We also observe that by graded-commutativity of the Tate cohomology ring, every element of odd degree has square 0. In particular, $w_1 w_1 = -w_1 w_1$ implies $w_1^2 = 0$. One can check that $\widehat{H}^*(N)$ is an $\widehat{H}^*(G)$ -module with action via res_N^G :

$$x \cdot w_1 = w_1 w_2 w_1 = w_1^2 w_2 = 0,$$

$$x \cdot w_2 = w_1 w_2 w_2 = (-1)^2 w_2 w_1 w_2 = (-1)^2 w_2 w_2 w_1 = z \cdot w_1,$$

$$x \cdot w_2^{-1} = w_1 w_2 w_2^{-1} = w_1,$$

$$\begin{aligned}
z \cdot w_2 &= w_2^2 w_2 = w_2^3, \\
z \cdot w_2^{-1} &= w_2^2 w_2^{-1} = w_2, \\
z^{-1} \cdot w_1 &= w_2^{-1} w_2^{-1} w_1 = (-1)^{-2} w_2^{-1} w_1 w_2^{-1} = (-1)^{-2} w_1 w_2^{-1} w_2^{-1} = (xz^{-1}) \cdot w_2^{-1}, \\
z^{-1} \cdot w_2 &= w_2^{-1} w_2^{-1} w_2 = w_2^{-1}, \\
z^{-1} \cdot w_2^{-1} &= (w_2^{-1})^3.
\end{aligned}$$

Therefore, as an $\widehat{H}^*(G)$ -module, $\widehat{H}^*(N)$ is generated by $1, w_1, w_2$ and w_2^{-1} , subject to the relation $x \cdot w_1 = 0$, $x \cdot w_2 = z \cdot w_1$, and $z^{-1} \cdot w_1 = (xz^{-1}) \cdot w_2^{-1}$. By the isomorphism in Lemma 5.3 and mapping through ψ_2 , we obtain the first line of the relations.

To check the second and third lines of the relations, we recall the fact that the submodule of invariants $(\mathbf{k}G)^G$ is the center $Z(\mathbf{k}G)$ of the group algebra $\mathbf{k}G$, which is generated by conjugacy class representatives of G . Therefore, we may identify the degree-0 Tate-Hochschild cohomology with a quotient of $Z(\mathbf{k}G)$, as $\widehat{HH}^0(\mathbf{k}G, \mathbf{k}G) \cong \widehat{H}^0(G, \mathbf{k}G)$ is a quotient of $H^0(G, \mathbf{k}G)$. Under this identification, E_i corresponds to (a quotient of) the sum of the group elements conjugate to g_i . In particular,

$$E_2^2 = (a + a^{-1})^2 = a^2 + 2 + a^{-2} = a^{-1} - 1 + a = E_2 - 1$$

in characteristic 3, which implies

$$C^2 = (E_2 + 1)^2 = E_2^2 + 2E_2 + 1 = 3E_2 = 0.$$

For the rest of the relations, we use the product formula in Theorem 5.5. Let α and β be elements of $\widehat{H}^*(N)$, we have:

$$\psi_2(\alpha) \smile \psi_2(\beta) = \psi_2(b^*(\alpha\beta)) + \psi_1(\text{cor}_N^G(\alpha b^*(\beta))).$$

Recall that $b^* : \widehat{H}^*(N) \rightarrow \widehat{H}^*(bN) = \widehat{H}^*(N)$. By checking on the definition of b^* and the degrees of w_i , we see that $b^*(w_2^{-1}) = -w_2^{-1}$ and $b^*(w_i) = -w_i$, for $i = 1, 2$. Moreover, as there are no degree-1, 2 and -2 elements in $\widehat{H}^*(G)$, we have $\text{cor}_N^G(w_1) = \text{cor}_N^G(w_2) = \text{cor}_N^G(w_2^{-1}) = 0$. Similarly, by checking on the cochain level and using Lemma 4.3 (10), for all $n \in \mathbb{Z}$, we obtain:

$$\text{cor}_N^G(w_2^n) = \begin{cases} 0, & n \text{ is odd} \\ -z^{n/2}, & n \text{ is even.} \end{cases}$$

Hence, using Lemma 4.3 (10) again,

$$\text{cor}_N^G(w_1 w_2^n) = \begin{cases} -xz^{(n-1)/2}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

Let $\alpha = 1$ and $\beta = w_1$, using the product formula in Theorem 5.5, we obtain:

$$\psi_2(1) \smile \psi_2(w_1) = \psi_2(b^*(w_1)) + \psi_1(\text{cor}_N^G(b^*(w_1))) = \psi_2(-w_1) + 0 = -W_1.$$

So $CW_1 = (E_2 + 1)W_1 = E_2W_1 + W_1 = -W_1 + W_1 = 0$. Similarly, let $\alpha = 1$ and $\beta = w_2$ or w_2^{-1} , we show that $CW_2 = 0 = CW_2^{-1}$. This proves the second line of the relations.

Let $\alpha = \beta = w_2$, we have:

$$\begin{aligned}
W_2^2 &= \psi_2(w_2) \smile \psi_2(w_2) = \psi_2(b^*(w_2^2)) + \psi_1(\text{cor}_N^G(w_2 b^*(w_2))) \\
&= \psi_2(\text{res}_N^G z \smile 1) + \psi_1(z) \\
&= z \smile \psi_2(1) + z
\end{aligned}$$

$$= zE_2 + z = zC.$$

Similarly, for $\alpha = \beta = w_2^{-1}$, we acquire that $W_2^{-2} = z^{-1}C$.

Let $\alpha = w_1$ and $\beta = w_2^{-1}$:

$$\begin{aligned} W_1W_2^{-1} &= \psi_2(w_1) \smile \psi_2(w_2^{-1}) = \psi_2(b^*(w_1w_2^{-1})) + \psi_1(\text{cor}_N^G(w_1b^*(w_2^{-1}))) \\ &= \psi_2(\text{res}_N^G xz^{-1} \smile 1) + \psi_1(xz^{-1}) \\ &= xz^{-1} \smile \psi_2(1) + xz^{-1} \\ &= xz^{-1}E_2 + xz^{-1} = xz^{-1}C. \end{aligned}$$

Using the same argument, for $\alpha = w_1$ and $\beta = w_2$, we obtain $W_1W_2 = xC$. Thus, we have found all necessary relations for the generators of $\widehat{\text{HH}}^*(\mathbf{k}G, \mathbf{k}G)$. Furthermore, because the ring $\widehat{\text{H}}^*(G, \mathbf{k}G)$ is graded-commutative, its nilpotent elements all lie in its radical. We observe that $C^2 = 0 = W_1^2$, $W_2^3 = W_2^2W_2 = zCW_2 = 0$, and $(W_2^{-1})^3 = W_2^{-2}W_2^{-1} = z^{-1}CW_2^{-1} = 0$. This implies that C, W_1, W_2 , and W_2^{-1} are contained in the radical of the Tate-Hochschild cohomology ring $\widehat{\text{HH}}^*(\mathbf{k}G, \mathbf{k}G)$. Consequently, modulo radicals, the algebra monomorphism $\psi_1 : \widehat{\text{H}}^*(S_3) \rightarrow \widehat{\text{HH}}^*(\mathbf{k}S_3, \mathbf{k}S_3)$ induces an isomorphism. \square

REFERENCES

- [1] Luchezar L. Avramov and Alex Martsinkovsky, "Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension," *Proc. London Math. Soc.* (3) **85** (2002), 393-440.
- [2] D. J. Benson and J. F. Carlson, "Products in negative cohomology," *J. Pure & Applied Algebra*, **82** (1992), 107-129.
- [3] P. A. Bergh and D. A. Jorgensen, "Tate-Hochschild homology and cohomology of Frobenius algebras," to appear in *Journal of Noncommutative Geometry*, (2011).
- [4] D. J. Benson, "Commutative Algebra in the Cohomology of Groups," *Trends in Commutative Algebra*, MSRI Publications, **51** (2004), 1-50.
- [5] D. J. Benson, *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, **30** (1991).
- [6] D. J. Benson, *Representations and Cohomology II: Cohomology of groups and modules*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, **31** (1991).
- [7] K. S. Brown, *Cohomology of Groups*, Graduate texts in mathematics, **87** Springer-Verlag, 1982.
- [8] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, NJ, 1956.
- [9] C. Cibils and A. Solotar, "Hochschild cohomology algebra of abelian groups," *Arch. Math.*, **68** (1997), 17-21.
- [10] V. C. Nguyen, "Tate and Tate-Hochschild Cohomology for finite dimensional Hopf Algebras," (submitted) arXiv:1209.4888.
- [11] S. F. Siegel and S. J. Witherspoon, "The Hochschild cohomology ring of a group algebra," *Proc. London Math. Soc.* **79** (1999), 131-157.
- [12] J. Tate, "The higher dimensional cohomology groups of class field theory," *Ann. of Math.* (2) **56** (1952), 294-297.

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